Rationality of conic bundle 3-folds over non-closed fields

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X sm proj var $/k$ - any field, $\dim X = n$.

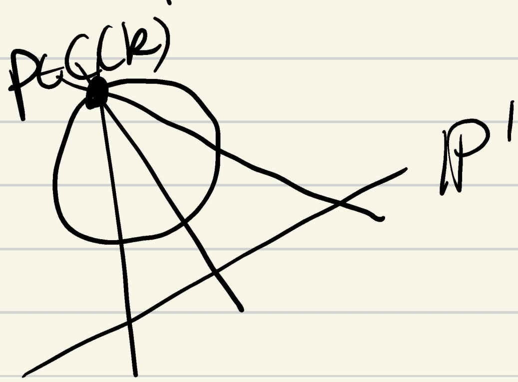
- X is <u>rational</u> if $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$ $/k$
- X is <u>geometrically rational</u> if $X_{\bar{k}} \overset{\sim}{\dashrightarrow} \mathbb{P}^n_{\bar{k}}$.

<u>Question</u>: When is a geometrically rational variety rational?

<u>Ex</u>: C curve $/k$ genus 0. C rational $\Leftrightarrow$ $C(k) \neq \emptyset$.

($\Rightarrow$) Lang-Nishimura lemma: existence of k-pt is a birational invariant

($\Leftarrow$)



projection from p gives rational parameterization.

To show rationality: give explicit construction

To show irrationality: use obstructions to rationality

$X \times \mathbb{P}^m \overset{\sim}{\dashrightarrow} \mathbb{P}^{n+m}$ $\qquad$ $\mathbb{P}^N \underset{\text{dominant}}{\dashrightarrow} X$

rational $\Longrightarrow$ stably rational $\Longrightarrow$ unirational

If $k = \mathbb{C}$, $\dim X = 1, 2$: unirational $\Rightarrow$ rational

Lüroth (1), Castelnuovo (2)

Counterexamples to reverse implications:

(Iskovskikh - Manin 71, Clemens - Griffiths 72) $k = \mathbb{C}$ unirational but not rational

(Artin - Mumford 72) $k = \mathbb{C}$ unirational but not stably rational

conic bundle 3-folds

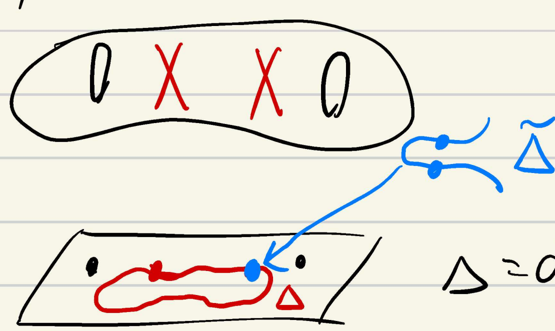
(BCTSSD 85) $k = \bar{k}$ stably rational but not rational

§ Conic bundles & rationality

char $k \neq 2$, S sm. surface

A <u>conic bundle</u> $\pi : X \to S$ is a flat fibration s.t. $X_\eta \subseteq \mathbb{P}^2_{k(S)}$ is a smooth conic.

$$X \downarrow S$$



discriminant double cover $\varpi : \tilde{\Delta} \to \Delta$

Δ = discriminant curve

Today, restrict to:

- X, S sm. proj.
- $\tilde{\Delta}, \Delta$ smooth & geom. irred. (geom. ordinary)

- $\rho(X_{\bar{k}}/S_{\bar{k}}) = 1$ (geom. standard)

for $X \to \mathbb{P}^2$ of this form:

- geom. rational $\Longleftrightarrow$ $\deg \Delta \leq 4$ (Iskovskikh) or $\deg \Delta = 5$ & $[\tilde{\Delta} \to \Delta]$ even theta characteristic (Panin) (char $= 0$)
- geom. irrational $\Longleftrightarrow$ $\deg \Delta \geq 6$ (Beauville) or $\deg \Delta = 5$ & $[\tilde{\Delta} \to \Delta]$ odd theta char. (Shokurov).

If $X \to \mathbb{P}^2$ is geom. rat'l:

- $\deg \Delta \leq 3$ & $X(k) \neq \emptyset$ $\Longrightarrow$ rational (Iskovskikh + ε)
- $\deg \Delta = 4$ & $\tilde{\Delta}(k) \neq \emptyset$ $\Longrightarrow$ rational (" ").

Focus: $\deg \Delta = 4$ & $\tilde{\Delta}(k) = \emptyset$.

Classical obstructions vanish:
- X unirational
- $\operatorname{Br} X = \operatorname{Br} \mathbb{P}^3 = \operatorname{Br} k$
- $\operatorname{Bir}(X)$ infinite
- Intermediate Jacobian obstruction: (Clemens–Griffiths, Beauville, ACMV, Benoist–Wittenberg)

  Y a sm. proj. 3-fold. $\exists$ abelian variety $IJ(Y)$ s.t.

$$Y \text{ rational} \Longrightarrow IJ(Y) \cong \prod \operatorname{Pic}^0_{C_i/k}$$

New: IJ torsor obstruction

(Hassett-Tschinkel 2021; Benoist-Wittenberg): Y is a
$k=\mathbb{R}$, $k \subseteq \mathbb{C}$ — k - any

geom. rational 3-fold. For each algebraic curve class γ on Y, $\exists$ a torsor $IJ^{\gamma}(Y)$ over $IJ(Y)$.

Y rational w/ $IJ(Y) \cong \mathrm{Pic}^0_{\Gamma}$, $g(\Gamma) \geq 2$ $\Longrightarrow$ $\forall\, \gamma$, $IJ^{\gamma}(Y) \cong \mathrm{Pic}^i_{\Gamma}$ for some i.

(HT, BW) A smooth complete intersection of quadrics $X \subseteq \mathbb{P}^5$ is rational $\Longleftrightarrow$ IJT obstruction vanishes.
$\Longleftrightarrow$ X contains a line $/k$.

(Kuznetsov-Prokhorov) Use IJT obs. to characterize rationality for Fano 3-folds w/ $\rho(Y_{\bar{k}})=1$ (char 0).

Theorem (FJSVV): $\exists$ geometrically rational conic bundles $X \to \mathbb{P}^2 / \mathbb{R}$, $\deg \Delta = 4$, $X(\mathbb{R}) \neq \phi$, that are irrational over $\mathbb{R}$:
[Ex 1]: X has no IJT obstruction, but $X(\mathbb{R})$ disconnected.
[Ex 2]: X w/ $X(\mathbb{R}) \underset{\text{homeo}}{\sim} S^3$, but has an IJT obs.

Method of Pf:
- explicit description of $IJ^{\gamma}(X)$ ← alg. curve class
- particular models $Y \xrightarrow[2:1]{} \mathbb{P}^1 \times \mathbb{P}^2$ branched over (2,2)-divisor.

§ IJ(X) & $IJ^\gamma(X)$.

$\pi : X \to \mathbb{P}^2$ conic bundle, $\varpi : \tilde{\Delta} \xrightarrow[2:1 \text{ étale}]{} \Delta$ disc. cover

Prym variety $\mathrm{Prym}_{\tilde{\Delta}/\Delta} = (\ker \varpi_*)^0$

$\varpi_* : \mathrm{Pic}_{\tilde{\Delta}} \to \mathrm{Pic}_{\Delta}$

(Beauville 77) If $k = \bar{k}$, then

$$\underbrace{(CH^2(X))^0}_{\text{algebraically trivial codim 2 cycles mod rat'l equiv.}} \cong \mathrm{Prym}_{\tilde{\Delta}/\Delta}(k)$$

Moreover, $IJ(X) \cong \mathrm{Prym}_{\tilde{\Delta}/\Delta}$.

Def: The polarized Prym scheme of $\tilde{\Delta} \to \Delta$ is

$$\mathrm{PPrym}_{\tilde{\Delta}/\Delta} := \bigcup_m \{ D \in \mathrm{Pic}_{\tilde{\Delta}} : \varpi_* D \sim \mathcal{O}_\Delta(m) \}$$

- group scheme w/ id comp. = $\mathrm{Prym}_{\tilde{\Delta}/\Delta}$.

Theorem (FJSVV): $\exists$ G_k-equivariant surj. gp homomorphism

$$CH^2(X_{\bar{k}}) \twoheadrightarrow \mathrm{PPrym}_{\tilde{\Delta}/\Delta}(\bar{k})$$

inducing an isomorphism btwn the components $\underbrace{CH^2(X_{\bar{k}})^\gamma}_{\gamma \text{ alg. curve class}}$ w/ $\bar{k}$-pts of components of $\mathrm{PPrym}_{\tilde{\Delta}/\Delta}$.

Moreover, if X is geom. rat'l,

$$IJ^\gamma(X) \cong \text{component of } \mathrm{PPrym}_{\tilde{\Delta}/\Delta}.$$

Pf idea:

proper transform of X_Δ

$$\begin{tikzcd} S \arrow[r, hook, "j"] \arrow[d, "p"', "\mathbb{P}^1\text{-bundle}"'] \arrow[dr] & & X' := Bl_{s(\Delta)} X \arrow[d, "\varepsilon"] \\ \tilde{\Delta} \arrow[d, "\varpi"'] & X_\Delta \arrow[r] & X \arrow[d] \\ \Delta \arrow[ur, "\delta"] \arrow[rr, hook] & & \mathbb{P}^2 \end{tikzcd}$$

use

$$\begin{tikzcd} \mathrm{Pic}_{\tilde{\Delta}}(\bar{k}) \arrow[r, "p^*"] & \mathrm{Pic}_S(\bar{k}) \arrow[r, "j_*"] & CH^2(X'_{\bar{k}}) \arrow[d, "\varepsilon_*"] \\ & & CH^2(X_{\bar{k}}) \end{tikzcd}$$

& $p_* j^* \varepsilon^*$.

$\square$.

Remark: curve classes (& hence torsors!) completely determined by $\tilde{\Delta} \to \Delta$

BUT: $X \to \mathbb{P}^2$ determined by $\tilde{\Delta} \to \Delta$ and a class in $Br(k)[2]$.

§ Double cover model.

$\tilde{\Delta} \xrightarrow[\substack{2:1 \\ \text{étale}}]{} \Delta$, $\Delta \subseteq \mathbb{P}^2$ quartic

(Bruin 2008) $\exists$ quadrics $Q_1, Q_2, Q_3 \in k[u,v,w]$ s.t. $\Delta = \{Q_1 Q_3 - Q_2^2 = 0\} \subseteq \mathbb{P}^2$.

Define $Y \xrightarrow[2:1]{} \mathbb{P}^1 \times \mathbb{P}^2$ by $z^2 = \underbrace{t_0^2 Q_1 + 2 t_0 t_1 Q_2 + t_1^2 Q_3}_{(2,2)\text{-divisor in } \mathbb{P}^1 \times \mathbb{P}^2}$

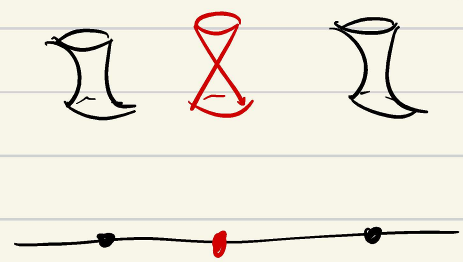
$$\begin{tikzcd}[column sep=small] & Y \arrow[dl, bend right] \arrow[d] \arrow[dr, bend left] & \\ \mathbb{P}^1 & \mathbb{P}^1\times\mathbb{P}^2 \arrow[l] \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

quadric surface bundle

conic bundle w/ discriminant cover $\tilde{\Delta} \to \Delta$

(Bruin) $\mathrm{Prym}_{\tilde{\Delta}/\Delta} \cong \mathrm{Pic}^0_\Gamma$ for Γ genus 2 curve:

$$\begin{tikzcd} F_1(Y/\mathbb{P}^1) \arrow[dr] & & Y \arrow[d] \\ & \Gamma \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$



Here, 4 distinct components of $\mathrm{PPrym}_{\tilde{\Delta}/\Delta}$:

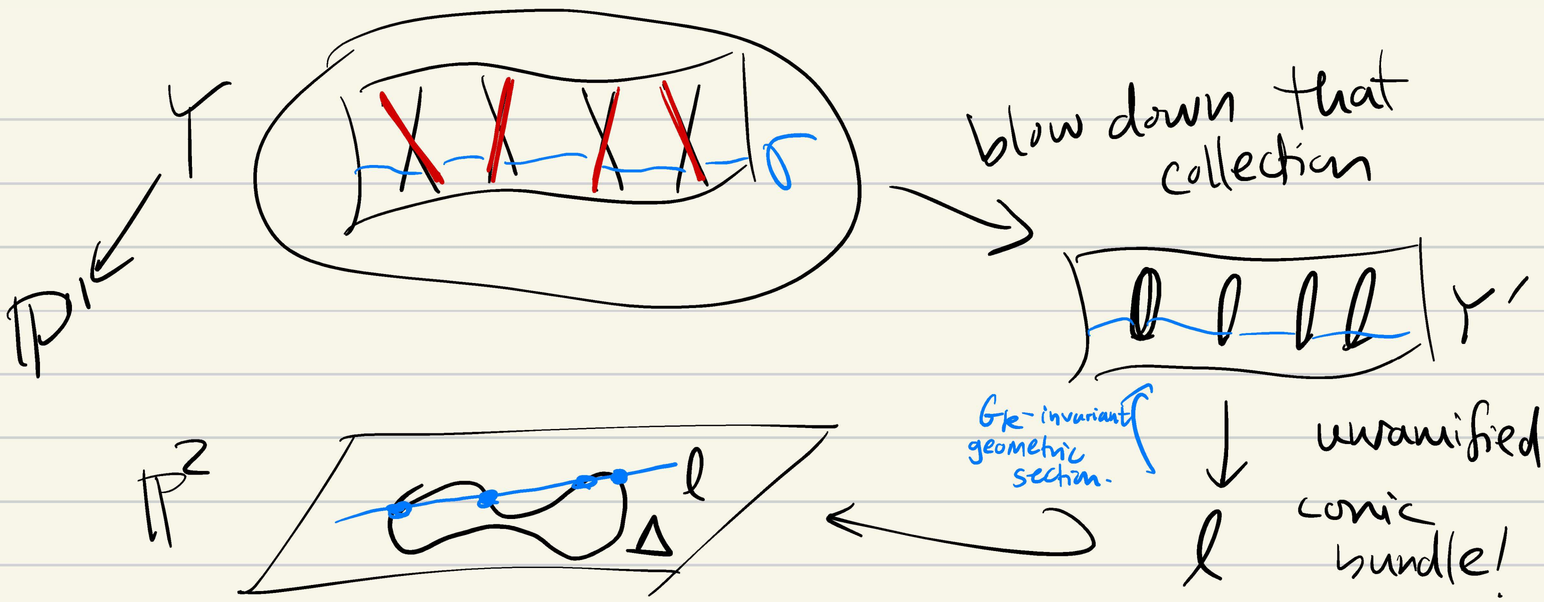
$P = \mathrm{Prym}_{\tilde{\Delta}/\Delta}$ $\tilde{P}$

$P^{(1)} = \ker \varpi_* \setminus P$ $\tilde{P}^{(1)}$ $\Big\}\ \{D : \varpi_* D \sim \mathcal{O}_\Delta(1)\}$

(Bruin) $P^{(1)} \cong \mathrm{Pic}^1_\Gamma$

Understanding $\tilde{P}^{(1)} = \{D : \varpi_* D \sim \mathcal{O}_\Delta(1),\ h^0(D) = 1\}$

$s \in \tilde{P}^{(1)}(k) \longleftrightarrow$ $\ell \subseteq \mathbb{P}^2$ line $+$ G_k-invariant choice of a component in each fiber of $Y|_\ell \to \ell$.


Y
P1
σ
blow down that collection
Y′
Gk-invariant geometric section.
unramified conic bundle!
ℓ
P2
ℓ
Δ